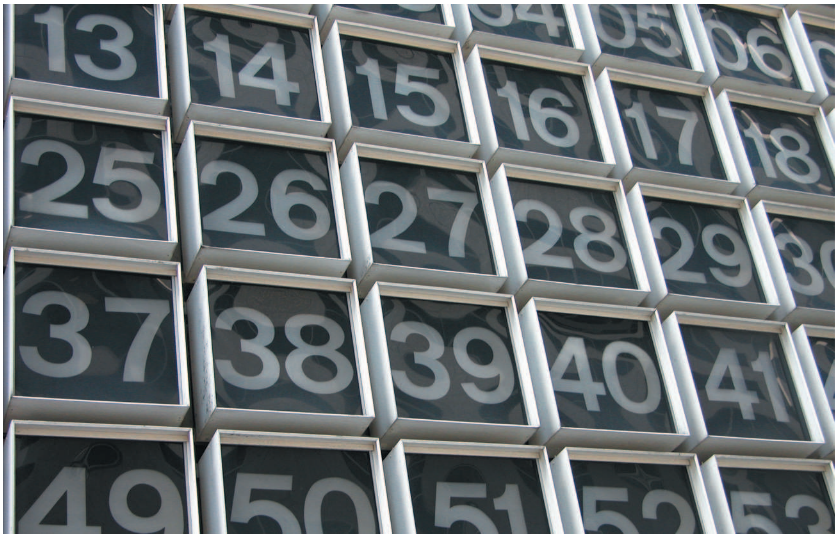
Suma kvadrata
prvih *n* uzastopnih prirodnih brojeva



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Ovaj prilog je nastavak članka istog naslova, objavljenog u MIŠ-u br. 59 (12. travnja 2011. g.). Dajemo još dva (ne tako uobičajena) dokaza.

Dokaz 3. Neka je $S_2 = 1^2 + 2^2 + 3^2 + \ldots + n^2$.

Zbog $1^2 = 1$, $2^2 = 1 + 3$, $3^2 = 1 + 3 + 5$, ..., $n^2 = 1 + 3 + 5 + \ldots + (2n - 1)^*$ je

$$\begin{aligned} S_2 &= 1 + (1+3) + (1+3+5) + \ldots + (1+3+5+\ldots+(2n+1)) \\ &= n \cdot 1 + (n-1) \cdot 3 + (n-2) \cdot 5 + \ldots + 1 \cdot (2n-1) \\ &= n[1+3+5+\ldots+(2n-1)] - [1 \cdot 3 + 2 \cdot 5 + \ldots + (n-1)(2n-1)] \end{aligned}$$

Izraz u drugoj uglatoj zagradi $1 \cdot 3 + 2 \cdot 5 + \ldots + (n-1)(2n-1)$ možemo pisati u obliku

$$(2 \cdot 1^2) + (2 \cdot 2^2 + 2) + \ldots + [2(n-1)^2 + (n-1)],$$

* Dokazat ćemo da je $1 + 3 + 5 + \ldots + (2n-1) = n^2$. Imamo:

$$\sum_{i=1}^{n}(2i-1) = 2 \cdot \sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = 2 \cdot \frac{n(n+1)}{2} - n = n^2$$

ili ovako

$$\begin{aligned} 1+3+5+\ldots+(2n-1) &= 1 + (2+1) + (3+2) + \ldots + [n + (n-1)] \\ &= (1+2+3+\ldots+n) + [1+2+3+\ldots+(n-1)] \\ &= \frac{n(n+1)}{2} + \frac{(n-1) \cdot n}{2} = n^2 \end{aligned}$$

pa sada imamo:

$$\begin{aligned}
S_2 &= n \cdot n^2 - \{(2 \cdot 1^2 + 1) + (2 \cdot 2^2 + 2) + \ldots + [2(n-1)^2 + (n-1)]\} \\
&= n^3 - \{\underbrace{[1 + 2 + 3 + \ldots + (n-1)]}_{=S_1 - n} + 2\underbrace{[1^2 + 2^2 + \ldots + (n-1)^2]}_{=S_2 - n^2}\} \\
&= n^3 - (S_1 - n) - 2(S_2 - n^2) = \left(\text{zbog } S_1 = \frac{n(n+1)}{2}\right) \\
&= n^3 - \frac{n(n+1)}{2} + n - 2S_2 + 2n^2 \\
&\iff 3 \cdot S_2 = n^3 - \frac{n^2}{2} - \frac{n}{2} + n + 2n^2 \\
&\iff S_2 = \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

Dokaz 4. Imamo da je

$$\begin{aligned}
S_2 &= 1^2 + 2^2 + 3^2 + \ldots + n^2 \\
&= 1 + (2+2) + (3+3+3) + \ldots + (n + n + \ldots + n) \\
&= (1 + 2 + \ldots + n) + (2 + 3 + 4 + \ldots + n) + (3 + 4 + 5 + \ldots + n) + \ldots + n.
\end{aligned}$$

Iz $\sum_{i=k}^{n} i = \sum_{i=1}^{n} i - \sum_{i=1}^{k-1} i$ proizlazi

$$\begin{aligned}
k + (k+1) + (k+2) + \ldots + n &= (1 + 2 + 3 + \ldots + n) - [1 + 2 + 3 + \ldots + (k-1)] \\
&= \frac{n(n+1)}{2} - \frac{(k-1) \cdot k}{2} = \frac{1}{2}(n^2 + n - k^2 + k),
\end{aligned}$$

pa je

$$\begin{aligned}
S_2 &= \frac{1}{2} \sum_{i=1}^{n} (n^2 + n - i^2 + i) \\
&= \frac{1}{2} \sum_{i=1}^{n} (n^2 + n) - \frac{1}{2} \sum_{i=1}^{n} i^2 + \frac{1}{2} \sum_{i=1}^{n} i \\
&= \frac{1}{2}(n^2 + n) \sum_{i=1}^{n} 1 - \frac{1}{2} S_2 + \frac{1}{2} S_1 \\
&= \frac{1}{2}(n^2 + n) \cdot n - \frac{1}{2} S_2 + \frac{1}{2} \cdot \frac{n(n+1)}{2},
\end{aligned}$$

tj.

$$\frac{3}{2} S_2 = \frac{1}{2} n^3 + \frac{1}{2} n^2 + \frac{1}{4} n^2 + \frac{1}{4} n \iff S_2 = \frac{n(n+1)(2n+1)}{6}.$$